

# A mathematical look at coupling

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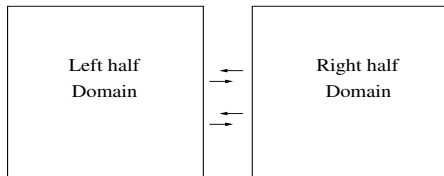
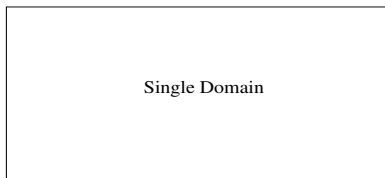
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# Outline

- ▶ Discretization consistent coupling: A way to exchange info between domains
- ▶ The mortar method: A way to handle misaligned cells on neighboring domains

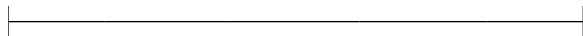
# Single domain vs. coupled domains: The continuous case

The continuous solution to the pde is the same, whether the problem is posed on a single domain, or on a split domain with appropriate matching conditions on the interface.



## Continuous equation on a single domain

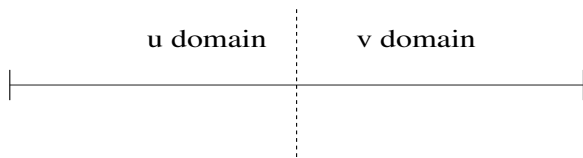
single domain



$$-z_{xx} = f(x)$$

$$z_x(0) = N \quad z(1) = D$$

## Continuous equation on a split domain with coupling



$$-u_{xx} = f(x)$$

$$u_x(0) = N$$

$$-v_{xx} = f(x)$$

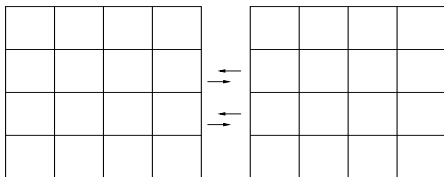
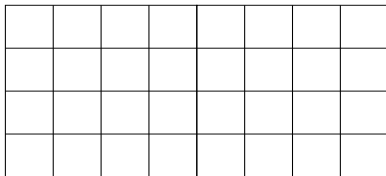
$$v(1) = D$$

$$u\left(\frac{1}{2}\right) = v\left(\frac{1}{2}\right)$$

$$v_x\left(\frac{1}{2}\right) = u_x\left(\frac{1}{2}\right)$$

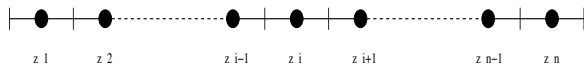
# Single domain vs. coupled domains: The discrete case

Are the two solutions the same on the discrete level?



With discretization consistent coupling, the two solutions are identical

# Discrete equations on a single domain



Left boundary (Neumann):

$$\frac{z_1 - z_2}{h} = F_1 - N$$

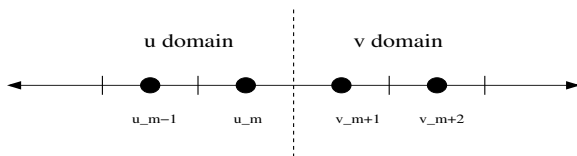
Interior:

$$\frac{-z_{i-1} + 2z_i - z_{i+1}}{h} = F_i$$

Right boundary (Dirichlet):

$$\frac{-z_{n-1} + 3z_n}{h} = F_n + \frac{2D}{h}$$

# Discrete equations on the u-domain



Left boundary (Neumann data):

$$\frac{u_1 - u_2}{h} = F_1 - N$$

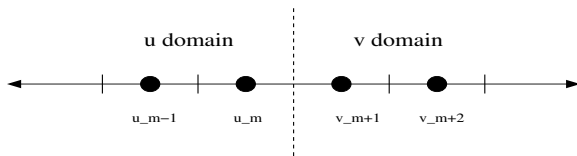
Interior:

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h} = F_i$$

Interface (No data):

$$\frac{-u_{m-1} + 3u_m}{h} = F_m + \frac{2v(\frac{1}{2})}{h} \quad \text{or} \quad \frac{-u_{m-1} + u_m}{h} = F_m + v_x(\frac{1}{2})$$

# Discrete equations on the v-domain



Interface (No data):

$$\frac{v_{m+1} - v_{m+2}}{h} = F_{m+1} - u_x\left(\frac{1}{2}\right) \quad \text{or} \quad \frac{3v_{m+1} - v_{m+2}}{h} = F_{m+1} + \frac{2u\left(\frac{1}{2}\right)}{h}$$

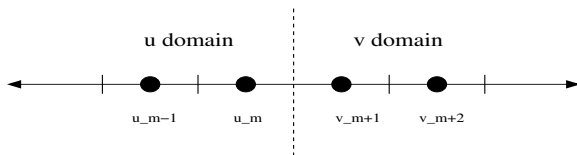
Interior:

$$\frac{-v_{i-1} + 2v_i - v_{i+1}}{h} = F_i$$

Right boundary (Dirichlet data):

$$\frac{-v_{n-1} + 3v_n}{h} = F_n + \frac{2D}{h}$$

# Discrete equations on a split domain with coupling



How to handle the equations at the interface?

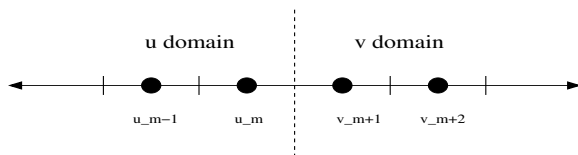
$$\frac{-u_{m-1} + 3u_m}{h} = F_m + \frac{2v(\frac{1}{2})}{h}$$

$$\frac{v_{m+1} - v_{m+2}}{h} = F_{m+1} - u_x(\frac{1}{2})$$

$$\frac{-u_{m-1} + u_m}{h} = F_m + v_x(\frac{1}{2})$$

$$\frac{3v_{m+1} - v_{m+2}}{h} = F_{m+1} + \frac{2u(\frac{1}{2})}{h}$$

## Geometric coupling



A geometric strategy uses an approximation for the unknown data in 2 of the 4 equations. The other 2 equations are not used.

$$\frac{-u_{m-1} + 3u_m}{h} = F_m + \frac{2v(\frac{1}{2})}{h} \quad \text{let } v(\frac{1}{2}) \approx v_{m+1}$$

$$\frac{v_{m+1} - v_{m+2}}{h} = F_{m+1} - u_x(\frac{1}{2}) \quad \text{let } u_x(\frac{1}{2}) \approx \frac{u_m - u_{m-1}}{h}$$

## Discretization consistent coupling

The discretization consistent strategy uses all 4 equations to eliminate the interface values rather than approximate them:

$$\frac{-u_{m-1} + 3u_m - 3v_{m+1} + v_{m+2}}{h} = F_m - F_{m+1}$$

$$\frac{-u_{m-1} + u_m + v_{m+1} - v_{m+2}}{h} = F_m + F_{m+1}$$

# Equivalence of discrete systems

The discretization consistent equations

$$\frac{-u_{m-1} + 3u_m - 3v_{m+1} + v_{m+2}}{h} = F_m - F_{m+1}$$

$$\frac{-u_{m-1} + u_m + v_{m+1} - v_{m+2}}{h} = F_m + F_{m+1}$$

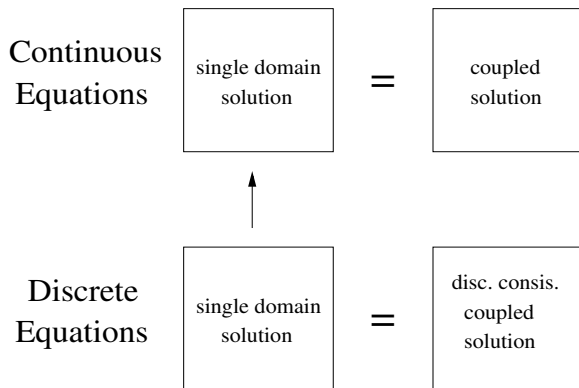
Are linear combinations of the single domain equations

$$\frac{-z_{m-1} + 2z_m - z_{m+1}}{h} = F_m$$

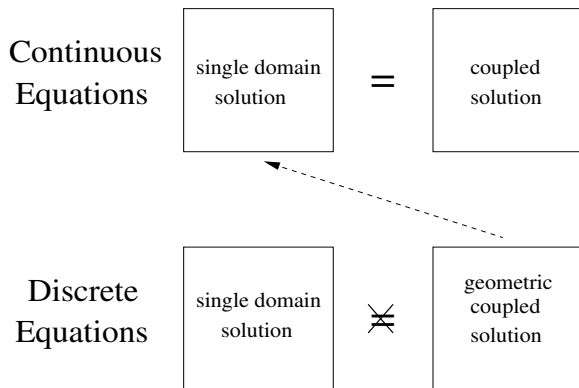
$$\frac{-z_m + 2z_{m+1} - z_{m+2}}{h} = F_{m+1}$$

So, the two solutions are identical

# Diagram of relationship between solutions

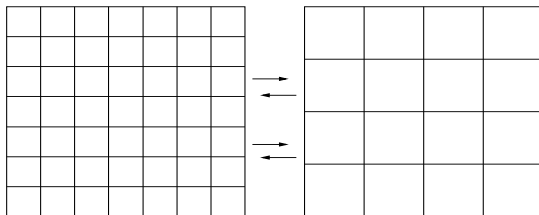
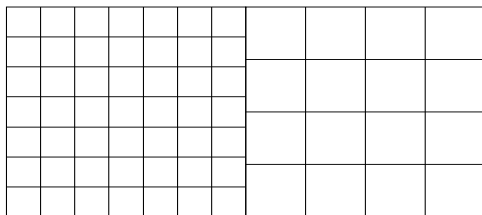


# Diagram of relationship between solutions

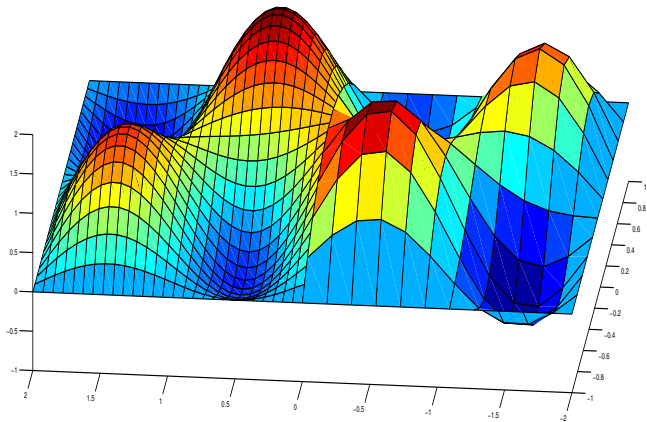


## Mortar method motivation: Misaligned grids

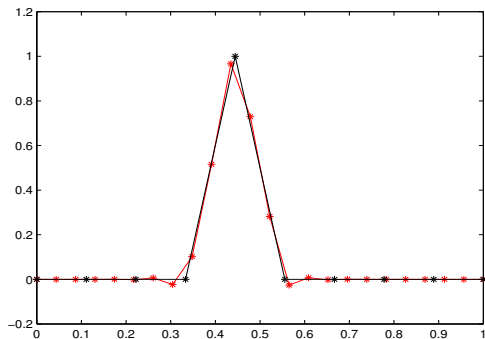
How to handle misaligned cells? Can we still exchange info?



# Mortar method example solution



# Mortar method matching condition



$$\int_{\Gamma} \left( \phi_i(x) - \sum_j q_{ij} \psi_j(x) \right) f_{\delta}(x) = 0$$

# Mortar method matching condition

$\phi_i(x)$  = trace of big side basis function along interface

$\eta_i(x) = \sum_j q_{ij} \psi_j(x)$  = trace of a linear combination of  
small side basis functions along interface

$f_\delta(x)$  = test space defined on interface

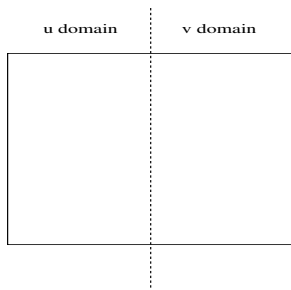
## 2D finite element example

$$-\nabla \cdot (\nabla u(x, y)) = f(x, y)$$

$$u(x, y) = 0 \quad \partial\Omega_u \setminus \Gamma$$

$$-\nabla \cdot (\nabla v(x, y)) = f(x, y)$$

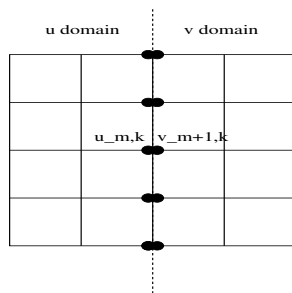
$$v(x, y) = 0 \quad \partial\Omega_v \setminus \Gamma$$



$$u\left(\frac{1}{2}, y\right) = v\left(\frac{1}{2}, y\right)$$

$$\hat{n} \cdot \nabla u\left(\frac{1}{2}, y\right) = \hat{n} \cdot \nabla v\left(\frac{1}{2}, y\right)$$

# Discretization consistent coupling for finite elements



$$u_{m,k} = u\left(\frac{1}{2}, y_k\right) \quad v_{m+1,k} = v\left(\frac{1}{2}, y_k\right)$$

$$\sum_{i,j} \langle \phi'_{i,j}, \phi'_{m,k} \rangle u_{j,k} = \langle f, \phi_{m,k} \rangle + \int_{\Gamma} (\hat{n} \cdot \nabla u) \phi_{m,k}$$

$$\sum_{i,j} \langle \eta'_{i,j}, \eta'_{m+1,k} \rangle v_{j,k} = \langle f, \eta_{m+1,k} \rangle - \int_{\Gamma} (\hat{n} \cdot \nabla v) \eta_{m+1,k}$$

## Other discretizations and models

Discretization consistent coupling has been implemented for

- ▶ Finite volume in 1D and 2D
- ▶ Standard finite elements in 1D and 2D
- ▶ Mortar finite elements in 2D
- ▶ A nonlinear heat equation (linear critical gradient model) in 1D

In each case the coupled discrete solution is identical to the single domain discrete solution

# Conclusion

- ▶ Discretization consistent coupling is a way to exchange info between domains that results in a coupled discrete solution that is identical to the single domain discrete solution.
- ▶ The mortar method is a way to handle misaligned cells on neighboring domains that is compatible with discretization consistent coupling.